Laplace transforms 1

Frames 1 to 93

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the Laplace transforms of simple standard expressions
- Use the first shift theorem to find the Laplace transform of a simple expression multiplied by an exponential
- Find the Laplace transform of a simple expression multiplied or divided by a variable
- Use partial fractions to find the inverse Laplace transform
- Use the 'cover up' rule
- Use the Laplace transforms of derivatives to solve differential equations
- Use the Laplace transform to solve simultaneous differential equations

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 26 Introduction to Laplace transforms

Introduction

1

The solution of a linear, ordinary differential equation with constant coefficients such as the second-order equation

$$af''(t) + bf'(t) + cf(t) = g(t)$$

can be solved by first obtaining the general form for the expression $f(t)$. This general form will contain a number of integration constants whose values can be found by applying the appropriate boundary conditions (see *Engineering Mathematics, Fifth Edition*, Programme 25). A more systematic way of solving such equations is to use the Laplace transform which converts the differential equation into an algebraic equation and has the added advantage of incorporating the boundary conditions from the beginning. Furthermore, in situations where $f(t)$ represents a function with discontinuities, the Laplace transform method can succeed where other methods fail.

Laplace transform techniques also provide powerful tools in numerous fields of technology such as Control Theory where a knowledge of the system transfer function is essential and where the Laplace transform comes into its own. Let us see what it is all about. (For a more detailed introduction see *Engineering Mathematics, Fifth Edition*, Programme 26.)

Laplace transforms

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt \qquad (1)$$

The parameter s is assumed to be positive and large enough to ensure that the integral converges. In more advanced applications s may be complex and in such cases the real part of s must be positive and large enough to ensure convergence.

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for t, so that the result will be an expression in s. Therefore

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

Make a note of this general definition: then we can apply it

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So we have $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t = F(s)$

Example 1

To find the Laplace transform of $f(t) = a$ (constant).

$$L\{a\} = \int_0^\infty ae^{-st}\mathrm{d}t = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty$$
$$= -\frac{a}{s}\{0-1\} = \frac{a}{s}$$
$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0) \qquad (2)$$

Example 2

To find the Laplace transform of $f(t) = e^{at}$ (a constant). As with all cases, we multiply $f(t)$ by e^{-st} and integrate between $t = 0$ and $t = \infty$.

$$\therefore\ L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t$$
$$= \dots\dots\dots\dots$$

Finish it off.

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$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

Because

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$
$$= -\frac{1}{s-a}\{0-1\} = \frac{1}{s-a}$$
$$\therefore\ L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a) \qquad (3)$$

So we already have two standard transforms

$$L\{a\} = \frac{a}{s} \quad \text{and} \quad L\{e^{at}\} = \frac{1}{s-a}$$

$$\therefore\ L\{4\} = \dots\dots\dots\dots; \qquad L\{e^{4t}\} = \dots\dots\dots\dots$$
$$L\{-5\} = \dots\dots\dots\dots; \qquad L\{e^{-2t}\} = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{ll} L\{4\} = \frac{4}{s}; & L\{e^{4t}\} = \frac{1}{s-4} \\ L\{-5\} = -\frac{5}{s}; & L\{e^{-2t}\} = \frac{1}{s+2} \end{array}}$$

Note that, as we said earlier, the Laplace transform is always an expression in s.

Now for some more examples

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Example 3

To find the Laplace transform of $f(t) = \sin at$. We could, of course, apply the definition and evaluate

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,\mathrm{d}t$$

using integration by parts.

However, it is much shorter if we use the fact that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

so that $\sin\theta$ is the imaginary part of $e^{j\theta}$, written $\mathscr{I}(e^{j\theta})$.

The function $\sin at$ can therefore be written $\mathscr{I}(e^{jat})$ so that

$$\begin{aligned} L\{\sin at\} = L\{\mathscr{I}(e^{jat})\} &= \mathscr{I}\int_0^\infty e^{jat}e^{-st}\,\mathrm{d}t = \mathscr{I}\int_0^\infty e^{-(s-ja)t}\,\mathrm{d}t \\ &= \mathscr{I}\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty\right\} = \mathscr{I}\left\{-\frac{1}{(s-ja)}[0-1]\right\} \\ &= \mathscr{I}\left\{\frac{1}{s-ja}\right\} \end{aligned}$$

We can rationalise the denominator by multiplying top and bottom by ....

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$$\boxed{s + ja}$$

$$\therefore\ L\{\sin at\} = \mathscr{I}\left\{\frac{s+ja}{s^2+a^2}\right\} = \frac{a}{s^2+a^2}$$

$$\therefore\ L\{\sin at\} = \frac{a}{s^2+a^2} \qquad (4)$$

We can use the same method to determine $L\{\cos at\}$ since $\cos at$ is the real part of e^{jat}, written $\Re(e^{jat})$.

Then $L\{\cos at\} = \ldots\ldots\ldots\ldots$

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$$\boxed{L\{\cos at\} = \frac{s}{s^2 + a^2}} \qquad (5)$$

Because

$$L\{\cos at\} = \Re\left\{\frac{s + ja}{s^2 + a^2}\right\} = \frac{s}{s^2 + a^2}$$

Recapping then: $L\{1\} = \dots\dots\dots\dots;$ $L\{e^{3t}\} = \dots\dots\dots\dots$
$L\{\sin 2t\} = \dots\dots\dots\dots;$ $L\{\cos 4t\} = \dots\dots\dots\dots$

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$$L\{1\} = \frac{1}{s}; \qquad L\{e^{3t}\} = \frac{1}{s-3}$$
$$L\{\sin 2t\} = \frac{2}{s^2 + 4}; \qquad L\{\cos 4t\} = \frac{s}{s^2 + 16}$$

Example 4

To find the transform of $f(t) = t^n$ where n is a positive integer.

By the definition $L\{t^n\} = \int_0^\infty t^n e^{-st}\,\mathrm{d}t.$

Integrating by parts

$$L\{t^n\} = \left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,\mathrm{d}t$$
$$= -\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t$$

We said earlier that in a product such as $t^n e^{-st}$ the numerical value of s is large enough to make the product converge to zero as $t \to \infty$

$$\therefore \left[t^n e^{-st}\right]_0^\infty = 0 - 0 = 0$$

$$\therefore L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t \qquad (6)$$

You will notice that $\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t$ is identical to $\int_0^\infty t^n e^{-st}\,\mathrm{d}t$ except that n is replaced by $(n-1)$.

$$\therefore \text{ If } I_n = \int_0^\infty t^n e^{-st}\,\mathrm{d}t, \text{ then } I_{n-1} = \int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t$$

and the result (6) becomes $I_n = \frac{n}{s}.I_{n-1}$ (7)

This is a reduction formula, and if we now replace n by $(n-1)$ we get

$I_{n-1} = \dots\dots\dots\dots$

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$$\boxed{I_{n-1} = \frac{n-1}{s}.I_{n-2}}$$

If we replace n by $(n-1)$ again in this last result, we have

$$I_{n-2} = \frac{n-2}{s}.I_{n-3}$$

$$\text{So } I_n = \int_0^\infty t^n e^{-st}\,dt = \frac{n}{s}.I_{n-1}$$

$$= \frac{n}{s}.\frac{n-1}{s}.I_{n-2}$$

$$= \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.I_{n-3} \quad \text{etc.}$$

$$= \dots\dots\dots\dots \text{ (next line)}$$

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$$\boxed{I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}.I_{n-4}}$$

So finally, we have

$$I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}\dots\frac{n-(n-1)}{s}.I_0$$

But $\quad I_0 = L\{t^0\} = L\{1\} = \dfrac{1}{s}$

$$\therefore\ I_n = \frac{n(n-1)(n-2)(n-3)\dots(3)(2)(1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\therefore\ L\{t^n\} = \frac{n!}{s^{n+1}} \qquad (8)$$

$$\therefore\ L\{t\} = \frac{1}{s^2};\quad L\{t^2\} = \frac{2}{s^3};\quad L\{t^3\} = \frac{6}{s^4}$$

and with $n = 0$, since $0! = 1$, the general result includes $L\{1\} = \dfrac{1}{s}$ which we have already established.

Example 5

Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Starting from the exponential definitions of $\sinh at$ and $\cosh at$, i.e.

$$\sinh at = \tfrac{1}{2}(e^{at} - e^{-at}) \quad \text{and} \quad \cosh at = \tfrac{1}{2}(e^{at} + e^{-at})$$

we proceed as follows.

(a) $f(t) = \sinh at$. $\quad L\{\sinh at\} = \displaystyle\int_0^\infty \sinh at\; e^{-st}\,dt$

$$= \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\}dt$$

$$= \dots\dots\dots\dots$$

Complete it

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$$\boxed{L\{\sinh at\} = \frac{a}{s^2 - a^2}}$$

Because

$$\frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

$$\therefore\ L\{\sinh at\} = \frac{a}{s^2 - a^2} \qquad (9)$$

(b) $f(t) = \cosh at$. Proceeding in the same way

$$L\{\cosh at\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{\cosh at\} = \frac{s}{s^2 - a^2}}$$

$$L\{\cosh at\} = \frac{1}{2}\int_0^\infty (e^{at} + e^{-at})e^{-st}\, dt = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{s^2 - a^2}\right\} = \frac{s}{s^2 - a^2}$$

$$\therefore\ L\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad (10)$$

So we have accumulated several standard results:

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}; \qquad L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Make a note of this list if you have not already done so: it forms the basis of much that is to follow.

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The Laplace transform is a linear transform, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

Note: Two transforms must **not** be multiplied together to form the transform of a product of expressions – we shall see later that the product of two transforms is the transform of the *convolution* of two expressions.

Example 6

(a) $$L\{2e^{-t}+t\} = L\{2e^{-t}\} + L\{t\} = 2L\{e^{-t}\} + L\{t\} = \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2+s+1}{s^2(s+1)}$$

(b) $$L\{2\sin 3t + \cos 3t\} = 2L\{\sin 3t\} + L\{\cos 3t\} = 2.\frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9}$$

(c) $$L\{4e^{2t} + 3\cosh 4t\} = 4L\{e^{2t}\} + 3L\{\cosh 4t\} = 4.\frac{1}{s-2} + 3.\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16} = \frac{7s^2-6s-64}{(s-2)(s^2-16)}$$

So 1. $L\{2\sin 3t + 4\sinh 3t\} = \dots\dots\dots\dots$

2. $L\{5e^{4t} + \cosh 2t\} = \dots\dots\dots\dots$

3. $L\{t^3 + 2t^2 - 4t + 1\} = \dots\dots\dots\dots$

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1. $\dfrac{18(s^2+3)}{s^4-81}$; 2. $\dfrac{6s^2-4s-20}{(s-4)(s^2-4)}$; 3. $\dfrac{1}{s^4}\{s^3-4s^2+4s+6\}$

The working is straightforward.

1. $$L\{2\sin 3t + 4\sinh 3t\} = 2.\frac{3}{s^2+9} + 4.\frac{3}{s^2-9} = \frac{6}{s^2+9} + \frac{12}{s^2-9} = \frac{18(s^2+3)}{s^4-81}$$

2. $$L\{5e^{4t} + \cosh 2t\} = \frac{5}{s-4} + \frac{s}{s^2-4} = \frac{6s^2-4s-20}{(s-4)(s^2-4)}$$

▶

3. $L\{t^3 + 2t^2 - 4t + 1\} = \dfrac{3!}{s^4} + 2.\dfrac{2!}{s^3} - 4.\dfrac{1!}{s^2} + \dfrac{1}{s}$

$$= \frac{1}{s^4}\{s^3 - 4s^2 + 4s + 6\}$$

We have been building up a list of standard transforms of simple expressions. Before we leave this part of the work, there are three important and useful theorems which enable us to deal with rather more complicated expressions.

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Theorem 1 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then

$L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \displaystyle\int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

That is

$$L\{e^{-at}f(t)\} = F(s+a)$$

The transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

For example $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

then $L\{e^{-3t}\sin 2t\} = \dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

Similarly, $L\{t^2\} = \dfrac{2}{s^3}$ $\quad\therefore\ L\{t^2e^{4t}\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{2}{(s-4)^3}}$$

Because $L\{t^2\} = \dfrac{2}{s^3}$. $\therefore\ L\{t^2e^{4t}\}$ is the same with s replaced by $(s-4)$.

$$\therefore\ L\{t^2e^{4t}\} = \frac{2}{(s-4)^3}$$

Here is a short exercise by way of practice.

Exercise

Determine the following.

1. $L\{e^{-2t}\cosh 3t\}$
2. $L\{2e^{3t}\sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t}\cos t\}$
5. $L\{e^{3t}\sinh 2t\}$
6. $L\{t^3e^{-4t}\}$

Complete all six and then check with the results in the next frame

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Here they are.

1. $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$ $\quad \therefore\ L\{e^{-2t}\cosh 3t\} = \dfrac{s+2}{(s+2)^2 - 9} = \dfrac{s+2}{s^2 + 4s - 5}$

2. $L\{\sin 3t\} = \dfrac{3}{s^2 + 9}$ $\quad \therefore\ L\{2e^{3t}\sin 3t\} = \dfrac{6}{(s-3)^2 + 9} = \dfrac{6}{s^2 - 6s + 18}$

3. $L\{4t\} = 4.\dfrac{1}{s^2}$ $\quad \therefore\ L\{4te^{-t}\} = \dfrac{4}{(s+1)^2}$

4. $L\{\cos t\} = \dfrac{s}{s^2 + 1}$ $\quad \therefore\ L\{e^{2t}\cos t\} = \dfrac{s-2}{(s-2)^2 + 1} = \dfrac{s-2}{s^2 - 4s + 5}$

5. $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$ $\quad \therefore\ L\{e^{3t}\sinh 2t\} = \dfrac{2}{(s-3)^2 - 4} = \dfrac{2}{s^2 - 6s + 5}$

6. $L\{t^3\} = \dfrac{3!}{s^4}$ $\quad \therefore\ L\{t^3 e^{-4t}\} = \dfrac{6}{(s+4)^4}$

Now let us deal with the next theorem

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Theorem 2 Multiplying by *t* and t^n

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

Because $L\{tf(t)\} = \displaystyle\int_{t=0}^{\infty} tf(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)\left(-\frac{\mathrm{d}e^{-st}}{\mathrm{d}s}\right)\mathrm{d}t$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = -F'(s)$$

That is

$$L\{tf(t)\} = -F'(s)$$

For example, $\quad L\{\sin 2t\} = \dfrac{2}{s^2 + 4}$

$$\therefore\ L\{t\sin 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

and similarly, $\quad L\{t\cosh 3t\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{s^2+9}{(s^2-9)^2}}$$

Because $L\{t\cosh 3t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right) = -\dfrac{(s^2-9)-s(2s)}{(s^2-9)^2} = \dfrac{s^2+9}{(s^2-9)^2}$

We could, if necessary, take this a stage further and find $L\{t^2\cosh 3t\}$

$$L\{t^2\cosh 3t\} = L\{t(t\cosh 3t)\} = -\frac{d}{ds}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}$$
$$= \frac{2s(s^2+27)}{(s^2-9)^3}$$

Likewise, starting with $L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$L\{t\sin 4t\} = \ldots\ldots\ldots\ldots$ and $L\{t^2\sin 4t\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{8s}{(s^2+16)^2};\quad \frac{8(3s^2-16)}{(s^2+16)^3}}$$

applying $L\{tf(t)\} = -\dfrac{d}{ds}\{F(s)\}$ in each case.

Theorem 2 obviously extends the range of functions that we can deal with.

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Make a note of this in your record book

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Theorem 3 Dividing by t

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma$

provided $\underset{t\to 0}{Lim}\left(\dfrac{f(t)}{t}\right)$ exists. To demonstrate this we start from the right-hand side of the result

$$\begin{aligned}\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma &= \int_{\sigma=s}^{\infty}\left\{\int_{t=0}^{\infty} f(t)e^{-\sigma t}\,dt\right\}d\sigma\\ &= \int_{t=0}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\,d\sigma\,dt\\ &= \int_{t=0}^{\infty} f(t)\left\{\int_{\sigma=s}^{\infty} e^{-\sigma t}\,d\sigma\right\}dt\\ &= \int_{t=0}^{\infty} f(t)\frac{e^{-st}}{t}\,dt\\ &= L\left\{\frac{f(t)}{t}\right\}\end{aligned}$$

Notice the dummy variable σ. The end result is an expression in s which comes from the lower limit of the integral so the variable of integration, which is absorbed during the process of integration, is changed to σ. Notice also that we interchange the order of integration.

This rule is somewhat restricted in use, since it is applicable only if $\underset{t\to 0}{Lim}\left(\dfrac{f(t)}{t}\right)$ exists. In indeterminate cases, we use L'Hôpital's rule to find out. Let's try a couple of examples.

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Example 1

Determine $L\left\{\dfrac{\sin at}{t}\right\}$

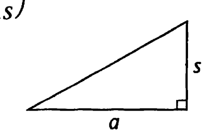
First we test $\underset{t\to 0}{Lim}\left\{\dfrac{\sin at}{t}\right\} = \left\{\dfrac{0}{0}\right\} = ?$ By L'Hôpital's rule, we differentiate top and bottom separately and substitute $t = 0$ in the result to ascertain the limit of the new expression.

$\underset{t\to 0}{Lim}\left\{\dfrac{\sin at}{t}\right\} = \underset{t\to 0}{Lim}\left\{\dfrac{a\cos at}{1}\right\} = a$, that is, the limit exists and the theorem can therefore be applied.

So $L\{\sin at\} = \dfrac{a}{s^2+a^2}$, therefore

$$\begin{aligned}L\left\{\frac{\sin at}{t}\right\} &= \int_s^{\infty}\frac{a}{\sigma^2+a^2}\,d\sigma\\ &= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^{\infty}\\ &= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)\\ &= \arctan\left(\frac{a}{s}\right)\end{aligned}$$

Notice that $\arctan\left(\dfrac{a}{s}\right) + \arctan\left(\dfrac{s}{a}\right) = \dfrac{\pi}{2}$, as can be seen from the figure



▶

Example 2

Determine $L\left\{\dfrac{1-\cos 2t}{t}\right\}$

First we test whether $\underset{t\to 0}{Lim}\left\{\dfrac{1-\cos 2t}{t}\right\}$ exists. Result? ....

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$$\boxed{\text{the limit exists}}$$

$$\underset{t\to 0}{Lim}\left\{\frac{1-\cos 2t}{t}\right\}=\frac{1-1}{0}=\frac{0}{0}=? \quad \therefore \text{ Apply l'Hôpital's rule.}$$

$$\underset{t\to 0}{Lim}\left\{\frac{1-\cos 2t}{t}\right\}=\underset{t\to 0}{Lim}\left\{\frac{2\sin 2t}{1}\right\}=\frac{0}{1}=0 \quad \therefore \text{ limit exists.}$$

$$L\{1-\cos 2t\}=\frac{1}{s}-\frac{s}{s^2+4}$$

Then, by Theorem 3

$$L\left\{\frac{1-\cos 2t}{t}\right\}=\int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma}-\frac{\sigma}{\sigma^2+4}\right\}d\sigma$$

$$=\left[\ln\sigma-\frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty}=\frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$

When $\sigma\to\infty$, $\ln\left(\dfrac{\sigma^2}{\sigma^2+4}\right)\to\ln 1=0$

Therefore, $L\left\{\dfrac{1-\cos 2t}{t}\right\}=$ ....

Complete it

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$$\boxed{\ln\sqrt{\frac{s^2+4}{s^2}}}$$

Because

$$L\left\{\frac{1-\cos 2t}{t}\right\}=-\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right)=\ln\left(\frac{s^2}{s^2+4}\right)^{-1/2}$$

$$=\ln\sqrt{\frac{s^2+4}{s^2}}$$

Let us pause here for a while and take stock, for we have met a number of results important in the future work.

▶

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$
a	$\dfrac{a}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
t^n	$\dfrac{n!}{s^{n+1}}$

(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

$$\text{If } L\{f(t)\} = F(s), \text{ then } L\{tf(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\{F(s)\}$$

4 *Theorem 3* Dividing by t

$$\text{If } L\{f(t)\} = F(s), \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$$

provided $\lim\limits_{t\to 0}\left\{\dfrac{f(t)}{t}\right\}$ exists.

Now let us work through a short revision exercise, so move on

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Exercise

Determine the Laplace transforms of the following expressions.

1	$\sin 3t$	**6**	$t\cosh 4t$
2	$\cos 2t$	**7**	$t^2 - 3t + 4$
3	e^{4t}	**8**	$\dfrac{e^{3t}-1}{t}$
4	$6t^2$	**9**	$e^{3t}\cos 4t$
5	$\sinh 3t$	**10**	$t^2 \sin t$

Complete the whole set and then check results with the next frame

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Here are the results.

1 $\dfrac{3}{s^2+9}$

2 $\dfrac{s}{s^2+4}$

3 $\dfrac{1}{s-4}$

4 $\dfrac{12}{s^3}$

5 $\dfrac{3}{s^2-9}$

6 $\dfrac{s^2+16}{(s^2-16)^2}$

7 $\dfrac{1}{s^3}(4s^2-3s+2)$

8 $\ln\left(\dfrac{s}{s-3}\right)$

9 $\dfrac{s-3}{s^2-6s+25}$

10 $\dfrac{6s^2-2}{(s^2+1)^3}$

It is just a case of applying the standard tranforms and the three theorems.

Now on to the next piece of work

Inverse transforms

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Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs.

For example, we know that $\dfrac{a}{s^2+a^2}$ is the Laplace transform of $\sin at$, so we can now write $L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$, the symbol L^{-1} indicating the inverse transform and **not** a reciprocal.

$\therefore$ (a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = \dots\dots\dots\dots$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = \dots\dots\dots\dots$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\} \dots\dots\dots\dots$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = \dots\dots\dots\dots$

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(a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = 4$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\} = \cos 5t$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = 4\sinh 3t$

Therefore, given a transform, we can write down the corresponding expression in t, provided we can recognise it from our table of transforms.

▶

But what about $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$? This certainly did not appear in our list of standard transforms.

In considering $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$, it happens that we can write $\frac{3s+1}{s^2-s-6}$ as the sum of two simpler functions $\frac{1}{s+2}+\frac{2}{s-3}$ which, of course, makes all the difference, since we can now proceed

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}$$

which we immediately recognise as ....

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$$\boxed{e^{-2t}+2e^{3t}}$$

The two simpler expressions $\frac{1}{s+2}$ and $\frac{2}{s-3}$ are called the *partial fractions* of $\frac{3s+1}{s^2-s-6}$, and the ability to represent a complicated algebraic fraction in terms of its partial fractions is the key to much of this work. Let us take a closer look at the rules.

Rules of partial fractions

1 The numerator must be of lower degree than the denominator. This is usually the case in Laplace transforms. If it is not, then we first divide out.

2 Factorise the denominator into its prime factors. These determine the shapes of the partial fractions.

3 A linear factor $(s+a)$ gives a partial fraction $\frac{A}{s+a}$ where A is a constant to be determined.

4 A repeated factor $(s+a)^2$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}$.

5 Similarly $(s+a)^3$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$.

6 A quadratic factor (s^2+ps+q) gives $\frac{Ps+Q}{s^2+ps+q}$.

7 Repeated quadratic factors $(s^2+ps+q)^2$ give

$$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

So $\frac{s-19}{(s+2)(s-5)}$ has partial fractions of the form ....

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$$\boxed{\frac{A}{s+2}+\frac{B}{s-5}}$$

and $\dfrac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fractions of the form ....

Be careful of the repeated factor.

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$$\boxed{\frac{A}{s+3}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}}$$

Let us work through the various steps with an example.

Example 1

To determine $L^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$.

(a) First we check that the numerator is of lower degree than the denominator. In fact, this is so.

(b) Factorise the denominator $\dfrac{5s+1}{s^2-s-12}=\dfrac{5s+1}{(s-4)(s+3)}$.

(c) Then the partial fractions are of the form ....

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$$\boxed{\frac{A}{s-4}+\frac{B}{s+3}}$$

We therefore have the identity

$$\frac{5s+1}{s^2-s-12}\equiv\frac{A}{s-4}+\frac{B}{s+3}$$

If we multiply through both sides by the denominator $s^2-s-12\equiv(s-4)(s+3)$ we have

$$5s+1\equiv A(s+3)+B(s-4)$$

This is also an identity and true for any value of s we care to substitute – our job is now to find the values of A and B.

We now substitute convenient values for s

(a) Let $(s-4)=0$, i.e. $s=4$ $\quad\therefore\ 21=A(7)+B(0)$ $\quad\therefore\ A=3$

(b) Let $(s+3)=0$, i.e. $s=-3$ and we get ....

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$$\boxed{B = 2}$$

$$\therefore \quad \frac{5s+1}{s^2-s-12} \equiv \frac{3}{s-4} + \frac{2}{s+3}$$

$$\therefore \; L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{3e^{4t} + 2e^{-3t}}$$

Example 2

Determine $L^{-1}\left\{\dfrac{9s-8}{s^2-2s}\right\}$.

Working as before, $f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{4 + 5e^{2t}}$$

Because

$$L\{f(t)\} = \frac{9s-8}{s^2-2s}.$$

(a) Numerator of first degree; denonominator of second degree. Therefore rule satisfied.

(b) $\dfrac{9s-8}{s(s-2)} \equiv \dfrac{A}{s} + \dfrac{B}{s-2}$.

(c) Multiply by $s(s-2)$. $\therefore\; 9s-8 = A(s-2) + Bs$.

(d) Put $s = 0$. $\quad -8 = A(-2) + B(0) \;\therefore\; A = 4$.

(e) Put $s - 2 = 0$, i.e. $s = 2$. $\quad 10 = A(0) + B(2) \;\therefore\; B = 5$.

$$\therefore\; f(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Example 3

Express $F(s) = \dfrac{s^2-15s+41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

$\dfrac{s^2-15s+41}{(s+2)(s-3)^2}$ has partial fractions of the form $\ldots\ldots\ldots\ldots$

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$$\boxed{\frac{A}{s+2}+\frac{B}{s-3}+\frac{C}{(s-3)^2}}$$

Now we multiply throughout by $(s+2)(s-3)^2$ and get

$s^2 - 15s + 41 \equiv A(s-3)^2 + B(s+2)(s-3) + C(s+2)$

Putting $(s-3) = 0$ and then $(s+2) = 0$ we obtain ....

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$$\boxed{A = 3 \text{ and } C = 1}$$

Now that we have run out of 'crafty' substitutions, we equate coefficients of the highest power of s on each side, i.e. the coefficients of s^2. This gives ....

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$$\boxed{1 = A + B \quad \therefore\ 1 = 3 + B \quad \therefore\ B = -2}$$

So $\dfrac{s^2-15s+41}{(s+2)(s-3)^2} = \dfrac{3}{s+2} - \dfrac{2}{s-3} + \dfrac{1}{(s-3)^2}$

Now $L^{-1}\left\{\dfrac{3}{s+2}\right\} = \ldots\ldots\ldots\ldots$ and $L^{-1}\left\{\dfrac{2}{s-3}\right\} = \ldots\ldots\ldots\ldots$

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$$\boxed{3e^{-2t} \text{ and } 2e^{3t}}$$

But what about $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$?

We remember that $L^{-1}\left\{\dfrac{1}{s^2}\right\} = \ldots\ldots\ldots\ldots$

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$$\boxed{t}$$

and that by Theorem 1, if $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$.

$\therefore\ \dfrac{1}{(s-3)^2}$ is like $\dfrac{1}{s^2}$ with s replaced by $(s-3)$ i.e. $a = -3$.

$$\therefore\ L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = te^{3t}$$

$$\therefore\ L^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\} = 3e^{-2t} + 2e^{3t} + te^{3t}$$

▶

Example 4

Determine $L^{-1}\left\{\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$.

Notice that this time we have a quadratic factor in the denominator

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)} \equiv \frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$\therefore\ 4s^2-5s+6 \equiv A(s^2+4)+(Bs+C)(s+1).$$

(a) Putting $(s+1)=0$, i.e. $s=-1$, $15=5A \quad \therefore\ A=3$

(b) Equate coefficients of highest power, i.e. s^2

$4=A+B \quad \therefore\ 4=3+B \quad \therefore\ B=1$

(c) We now equate the lowest power on each side, i.e. the constant term

$6=4A+C \quad \therefore\ 6=12+C \quad \therefore\ C=-6$

Now you can finish it off. $f(t)=$ ....

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$$\boxed{f(t)=3e^{-t}+\cos 2t-3\sin 2t}$$

Because

$$L\{f(t)\}=\frac{3}{s+1}+\frac{s}{s^2+4}-\frac{6}{s^2+4}$$

$$\therefore\ f(t)=3e^{-t}+\cos 2t-3\sin 2t$$

The 'cover up' rule

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While we can always find A, B, C, etc., there are many cases where we can use the 'cover up' methods and write down the values of the constant coefficients almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors. The following examples illustrate the method.

Example 1

We know that $F(s) = \dfrac{9s-8}{s(s-2)}$ has partial fractions of the form $\dfrac{A}{s} + \dfrac{B}{s-2}$.

By the 'cover up' rule, the constant A, that is the coefficient of $\dfrac{1}{s}$, is found by temporarily covering up the factor s in the denominator of $F(s)$ and finding the limiting value of what remains when s (the factor covered up) tends to zero.

Therefore A = coefficient of $\dfrac{1}{s} = \underset{s\to 0}{Lim}\left\{\dfrac{9s-8}{s-2}\right\} = 4$. That is $A = 4$.

Similarly, B, the coefficient of $\dfrac{1}{s-2}$, is obtained by covering up the factor $(s-2)$ in the denominator of $F(s)$ and finding the limiting value of what remains when $(s-2) \to 0$, that is $s \to 2$.

Therefore B = coefficient of $\dfrac{1}{s-2} = \underset{s\to 2}{Lim}\left\{\dfrac{9s-8}{s}\right\} = 5$. That is $B = 5$.

So that

$$\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{s-2}$$

Another example

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Example 2

$$F(s) = \frac{s+17}{(s-1)(s+2)(s-3)} \equiv \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

A: cover up $(s-1)$ in $F(s)$ and find

$$\underset{s\to 1}{Lim}\left\{\frac{s+17}{(s+2)(s-3)}\right\} = \frac{18}{-6} \quad \therefore\ A = -3$$

Similarly

B: .... $\therefore\ B =$ ....

C: .... $\therefore\ C =$ ....

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$$\boxed{\begin{array}{llll} B = \underset{s\to -2}{Lim}\left\{\dfrac{s+17}{(s-1)(s-3)}\right\} = \dfrac{15}{(-3)(-5)} & = 1 & & \therefore\ B = 1 \\ C = \underset{s\to 3}{Lim}\left\{\dfrac{s+17}{(s-1)(s+2)}\right\} = \dfrac{20}{(2)(5)} & = 2 & & \therefore\ C = 2 \end{array}}$$

$$\therefore\ F(s) = \frac{1}{s+2} + \frac{2}{s-3} - \frac{3}{s-1}$$

So $f(t) = e^{-2t} + 2e^{3t} - 3e^{t}$

Every entry in our table of standard transforms gives rise to a corresponding entry in a similar table of inverse transforms. Let us tabulate such a list.

45 Table of inverse transforms

$F(s)$	$f(t)$	
$\dfrac{a}{s}$	a	
$\dfrac{1}{s+a}$	e^{-at}	
$\dfrac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\dfrac{a}{s^2+a^2}$	$\sin at$	
$\dfrac{s}{s^2+a^2}$	$\cos at$	
$\dfrac{a}{s^2-a^2}$	$\sinh at$	
$\dfrac{s}{s^2-a^2}$	$\cosh at$	

Theorem 1

The first shift theorem can be stated as follows.

If $F(s)$ is the Laplace transform of $f(t)$ then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

Here is a short revision exercise.

Exercise

1 Find the inverse transforms of

(a) $\dfrac{1}{2s-3}$; (b) $\dfrac{5}{(s-4)^3}$; (c) $\dfrac{3s+4}{s^2+9}$.

2 Express in partial fractions

(a) $\dfrac{22s+16}{(s+1)(s-2)(s+3)}$; (b) $\dfrac{s^2-11s+6}{(s+1)(s-2)^2}$.

3 Determine

(a) $L^{-1}\left\{\dfrac{4s^2-17s-24}{s(s+3)(s-4)}\right\}$; (b) $L^{-1}\left\{\dfrac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$.

▶

1	(a) $\frac{1}{2}e^{3t/2}$;	(b) $5t^2e^{4t}$;	(c) $3\cos 3t + \frac{4}{3}\sin 3t$
2	(b) $\frac{1}{s+1} + \frac{4}{s-2} - \frac{5}{s+3}$;	(b) $\frac{2}{s+1} - \frac{1}{s-2} - \frac{4}{(s-2)^2}$	
3	(a) $2 + 3e^{-3t} - e^{4t}$;	(b) $2e^{3t} + 3\cos 2t + \frac{5}{2}\sin 2t$	

Solution of differential equations by Laplace transforms

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To solve a differential equation by Laplace transforms, we go through four distinct stages

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

We have spent some time finding the transforms of a variety of functions of t and the inverse transforms of functions of s, i.e. we have largely covered steps (a) and (d) of the above list. However, to write a differential equation in Laplace transforms, we must obtain the transforms of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.

Transforms of derivatives

Let $f'(t)$ denote the first derivative of $f(t)$ with respect to t,
$f''(t)$ denote the second derivative of $f(t)$ with respect to t, etc.

Then $L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt$ by definition.

Integrating by parts

$$L\{f'(t)\} = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty f(t)\{-se^{-st}\}\,dt$$

When $t \to \infty$, $e^{-st}f(t) \to$ ....

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$$\boxed{0}$$

Because s is positive and large enough to ensure that e^{-st} decays faster than any possible growth of $f(t)$.

$\therefore\ L\{f'(t)\} = -f(0) + sL\{f(t)\}$

Replacing $f(t)$ by $f'(t)$ gives

$$L\{f''(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

Because

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

so $\quad L\{f''(t)\} = -f'(0) + sL\{f'(t)\}$

$$= -f'(0) + s(-f(0) + sL\{f(t)\})$$

Writing $\quad L\{f(t)\} = F(s)$ as usual, we have

$$L\{f(t)\} = F(s)$$
$$L\{f'(t)\} = sF(s) - f(0)$$
$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

We can see a pattern emerging

$$L\{f'''(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'0) - f''(0)}$$

Alternative notation

We make the working neater by adopting the following notation.
Let $x = f(t)$ and at $t = 0$, we write

$$x = x_0 \qquad \text{i.e. } f(0) = x_0$$
$$\frac{dx}{dt} = x_1 \qquad \text{i.e. } f'(0) = x_1$$
$$\frac{d^2x}{dt^2} = x_2 \qquad \text{i.e. } f''(0) = x_2 \text{ etc.}$$
$$\therefore\ \frac{d^nx}{dt^n} = x_n \qquad \text{i.e. } f^n(0) = x_n$$

Also we denote the Laplace transform of x by $\bar{x}$,

i.e. $\bar{x} = L\{x\} = L\{f(t)\} = F(s)$.

So, using the 'dot' notation for derivatives, the previous results can be written ....

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$$\boxed{\begin{aligned} L\{x\} &= \bar{x} \\ L\{\dot{x}\} &= s\bar{x} - x_0 \\ L\{\ddot{x}\} &= s^2\bar{x} - sx_0 - x_1 \\ L\{\dddot{x}\} &= s^3\bar{x} - s^2x_0 - sx_1 - x_2 \end{aligned}}$$

In each case, the subscript indicates the order of the derivative, i.e. $x_n =$ the value of $\dfrac{d^n x}{dt^n}$ at $t = 0$.

Notice the pattern of the results.

$$L\{\ddddot{x}\} = \dots\dots\dots\dots$$

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$$\boxed{L\{\ddddot{x}\} = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3}$$

Now, at long last, we can start solving differential equations.

Solution of first-order differential equations

Example 1

Solve the equation $\dfrac{dx}{dt} - 2x = 4$ given that at $t = 0$, $x = 1$.

We go through the four stages.

(a) Rewrite the equation in Laplace transforms, using the last notation

$$L\{x\} = \bar{x};\quad L\{\dot{x}\} = \dots\dots\dots\dots$$
$$L\{4\} = \dots\dots\dots\dots$$

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$$\boxed{L\{\dot{x}\} = s\bar{x} - x_0;\quad L\{4\} = \frac{4}{s}}$$

Then the equation becomes $\quad (s\bar{x} - x_0) - 2\bar{x} = \dfrac{4}{s}$

(b) Insert the initial condition that at $t = 0$, $x = 1$, i.e. $x_0 = 1$

$$\therefore\ s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$$

(c) Now we rearrange this to give an expression for $\bar{x}$

$$\bar{x} = \dots\dots\dots\dots$$

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$$\bar{x} = \frac{s+4}{s(s-2)}$$

(d) Finally, we take inverse transforms to obtain x.

$\dfrac{s+4}{s(s-2)}$ in partial fractions gives ....

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$$\frac{3}{s-2} - \frac{2}{s}$$

Because

$\dfrac{s+4}{s(s-2)} \equiv \dfrac{A}{s} + \dfrac{B}{s-2}$ $\quad\therefore\ s+4 = A(s-2) + Bs$

(1) Put $(s-2) = 0$, i.e. $s = 2$ $\quad\therefore\ 6 = B(2)\quad \therefore\ B = 3$

(2) Put $s = 0$ $\quad\therefore\ 4 = A(-2)\quad \therefore\ A = -2$

$$\therefore\ \bar{x} = \frac{s+4}{s(s-2)} = \frac{3}{s-2} - \frac{2}{s}$$

Therefore, taking inverse transforms

$$x = L^{-1}\left\{\frac{s+4}{s(s-2)}\right\} = L^{-1}\left\{\frac{3}{s-2} - \frac{2}{s}\right\} = \text{...........}$$

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$$x = 3e^{2t} - 2$$

This solution should now be substituted back into the differential equation to verify that it is, indeed, correct.

Example 2

Solve the equation $\dfrac{dx}{dt} + 2x = 10e^{3t}$ given that at $t = 0$, $x = 6$.

(a) Convert the equations to Laplace transforms, i.e.

....

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$$(s\bar{x} - x_0) + 2\bar{x} = \frac{10}{s-3}$$

(b) Insert the initial condition, $x_0 = 6$

$$s\bar{x} - 6 + 2\bar{x} = \frac{10}{s-3}$$

(c) Rearrange to obtain $\bar{x} =$ ....

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$$\boxed{\bar{x} = \frac{6s-8}{(s+2)(s-3)}}$$

(d) Taking inverse transforms to obtain x

$$x = L^{-1}\left\{\frac{6s-8}{(s+2)(s-3)}\right\} = \dots\dots\dots\dots$$

Complete the solution

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$$\boxed{x = 4e^{-2t} + 2e^{3t}}$$

Because

$$\frac{6s-8}{(s+2)(s-3)} \equiv \frac{A}{s+2} + \frac{B}{s-3}$$

$$\therefore\ 6s - 8 = A(s-3) + B(s+2)$$

(1) Put $(s-3) = 0$, i.e. $s = 3$ $\quad\therefore\quad 10 = B(5) \quad\therefore\ B = 2$

(2) Put $(s+2) = 0$, i.e. $s = -2$. $\quad\therefore\quad -20 = A(-5) \quad\therefore\ A = 4$

$$\therefore\ \bar{x} = \frac{6s-8}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{2}{s-3}$$

$$\therefore\ x = L^{-1}\left\{\frac{4}{s+2} + \frac{2}{s-3}\right\} = 4e^{-2t} + 2e^{3t}$$

Example 3

Solve the equation $\dfrac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$, given that at $t = 0$, $x = 0$.

Work this through the four steps in the same way as before and complete it on your own.

$$x = \dots\dots\dots\dots$$

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$$\boxed{x = e^{4t} - e^{2t} + te^{4t}}$$

The working is quite standard.

$$\frac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$$

(a) $(s\bar{x} - x_0) - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

(b) $x_0 = 0 \quad \therefore\ s\bar{x} - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

(c) $\therefore\ \bar{x} = \dfrac{2}{(s-2)(s-4)} + \dfrac{1}{(s-4)^2}$

(d) $\dfrac{2}{(s-2)(s-4)} \equiv \dfrac{A}{s-2} + \dfrac{B}{s-4} \quad \therefore\ 2 = A(s-4) + B(s-2)$

Putting $(s-2) = 0$, i.e. $s = 2 \quad \therefore\ 2 = A(-2) \quad \therefore\ A = -1$

Putting $(s-4) = 0$, i.e. $s = 4 \quad \therefore\ 2 = B(2) \quad \therefore\ B = 1$

$\therefore\ \bar{x} = \dfrac{1}{s-4} - \dfrac{1}{s-2} + \dfrac{1}{(s-4)^2}$

$\therefore\ x = e^{4t} - e^{2t} + te^{4t}$

Now on to the next frame

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Solution of second-order differential equations

The method is, in effect, the same as before, going through the same four distinct stages.

Example 1

Solve the equation $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2e^{3t}$, given that at $t = 0$, $x = 5$ and $\dfrac{dx}{dt} = 7$.

(a) We rewrite the equation in terms of its transforms, remembering that

$L\{x\} = \bar{x}$

$L\{\dot{x}\} = s\bar{x} - x_0$

$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$

The equation becomes ....

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$$\boxed{(s^2\bar{x} - sx_0 - x_1) - 3(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}}$$

(b) Insert the initial conditions. In this case $x_0 = 5$ and $x_1 = 7$

$$\therefore\ (s^2\bar{x} - 5s - 7) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

(c) Rearrange to obtain $\bar{x} = \dots\dots\dots\dots$

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$$\boxed{\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}}$$

Because

$$s^2\bar{x} - 5s - 7 - 3s\bar{x} + 15 + 2\bar{x} = \frac{2}{s-3}$$

$$(s^2 - 3s + 2)\bar{x} - 5s + 8 = \frac{2}{s-3}$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s-3} + 5s - 8 = \frac{2 + 5s^2 - 23s + 24}{s-3}$$

$$\therefore\ \bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

(d) Now for partial fractions

$$\frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\therefore\ 5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

So that $A = \dots\dots\dots\dots$; $B = \dots\dots\dots\dots$; $C = \dots\dots\dots\dots$

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$$\boxed{A = 4;\quad B = 0;\quad C = 1}$$

$$\therefore\ \bar{x} = \frac{4}{s-1} + \frac{1}{s-3}$$

$$\therefore\ x = \dots\dots\dots\dots$$

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$$\boxed{x = 4e^t + e^{3t}}$$

As you see, the Laplace transform method can be considerably shorter than the classical method which requires

(a) determination of the complementary function
(b) determination of a particular integral
(c) obtaining the general solution, before
(d) arriving at the particular solution by substitution of the initial conditions in the general solution.

Here is another example.

Example 2

Solve $\dfrac{d^2x}{dt^2} - 4x = 24\cos 2t$ given that at $t = 0$, $x = 3$ and $\dfrac{dx}{dt} = 4$.

(a) In Laplace transforms ....

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$$\boxed{(s^2\bar{x} - sx_0 - x_1) - 4\bar{x} = \frac{24s}{s^2+4}}$$

(b) Insert initial condition, i.e. $x_0 = 3$; $x_1 = 4$

$$s^2\bar{x} - 3s - 4 - 4\bar{x} = \frac{24s}{s^2+4}$$

$$\therefore\ (s^2 - 4)\bar{x} = 3s + 4 + \frac{24s}{s^2+4}$$

$$= \frac{3s^3 + 4s^2 + 36s + 16}{s^2+4}$$

(c) $\bar{x} = \dfrac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)}$

Expressed in partial fractions, this becomes

....

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$$\boxed{\frac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)} \equiv \frac{As+B}{s^2+4} + \frac{C}{s-2} + \frac{D}{s+2}}$$

$$\therefore\ 3s^3 + 4s^2 + 36s + 16 \equiv (As+B)(s-2)(s+2) + C(s^2+4)(s+2) + D(s^2+4)(s-2)$$

Putting $(s-2) = 0$, i.e. $s = 2$, gives $C = 4$

Putting $(s+2) = 0$, i.e. $s = -2$, gives $D = 2$

Equating coefficients of s^3 and also the constant terms gives $A = -3$ and $B = 0$.

$$\therefore\ \bar{x} = \frac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)} = \frac{4}{s-2} + \frac{2}{s+2} - \frac{3s}{s^2+4}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$\boxed{x = 4e^{2t} + 2e^{-2t} - 3\cos 2t}$$

Now let us solve another equation, this time using the 'cover up' rule.

Example 3

Solve $\ddot{x} + 5\dot{x} + 6x = 4t$, given that at $t = 0$, $x = 0$ and $\dot{x} = 0$.

As usual we begin $(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 6\bar{x} = \dfrac{4}{s^2}$

$$x_0 = 0;\ x_1 = 0 \quad \therefore\ (s^2 + 5s + 6)\bar{x} = \frac{4}{s^2}$$

$$\therefore\ \bar{x} = \frac{4}{s^2(s+2)(s+3)}$$

The s^2 in the denominator can be awkward, so we introduce a useful trick and detach one factor s outside the main expression, thus

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{s(s+2)(s+3)}\right\} = \frac{1}{s}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right\}$$

Applying the 'cover up' rule to the expressions within the brackets

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{6}.\frac{1}{s} - \frac{2}{(s+2)} + \frac{4}{3}.\frac{1}{s+3}\right\}$$

Now we bring the external $\dfrac{1}{s}$ back into the fold

$$\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}.\frac{1}{s(s+3)}$$

and the second and third terms can be expressed in simple partial fractions so that

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+2} + \frac{4}{9}.\frac{1}{s} - \frac{4}{9}.\frac{1}{s+3}}$$

which can now be simplified into

$$\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{5}{9}.\frac{1}{s} + \frac{1}{s+2} - \frac{4}{9}.\frac{1}{s+3}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$x = \frac{2}{3}t - \frac{5}{9} + e^{-2t} - \frac{4}{9}e^{-3t}$$

There are times when a quadratic coefficient of $\bar{x}$ cannot be expressed in simple linear factors. In that case, we merely complete the square converting the expression into $(s \pm k)^2 \pm a^2$. Let us see such an example.

Example 4

Solve $\ddot{x} - 2\dot{x} + 10x = e^{2t}$, given that at $t = 0$, $x = 0$ and $\dot{x} = 1$.

We find the expression for $\bar{x}$ as before.

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\bar{x} = \frac{s-1}{(s-2)(s^2-2s+10)}$$

Because

$$(s^2\bar{x} - sx_0 - x_1) - 2(s\bar{x} - x_0) + 10\bar{x} = \frac{1}{s-2}$$

$$x_0 = 0;\ x_1 = 1 \qquad \therefore\ s^2\bar{x} - 1 - 2s\bar{x} + 10\bar{x} = \frac{1}{s-2}$$

$$\therefore\ (s^2 - 2s + 10)\bar{x} = 1 + \frac{1}{s-2} = \frac{s-1}{s-2}$$

$$\therefore\ \bar{x} = \frac{s-1}{(s-2)(s^2-2s+10)}$$

Expressing this in partial fractions

$\bar{x} = \ldots\ldots\ldots\ldots$ Evaluate the coefficients.

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$$\bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-10}{s^2-2s+10}\right\}$$

Because

$$\frac{s-1}{(s-2)(s^2-2s+10)} \equiv \frac{A}{(s-2)} + \frac{Bs+C}{s^2-2s+10}$$

$$\therefore\ s - 1 = A(s^2 - 2s + 10) + (s-2)(Bs + C)$$

Put $(s-2) = 0$, i.e. $s = 2$ $\quad 1 = A(4 - 4 + 10) \quad \therefore\ A = \frac{1}{10}$

$[s^2]$ $\quad 0 = A + B \quad \therefore\ B = -\frac{1}{10}$

[CT] $\quad -1 = 10A - 2C \quad \therefore\ 2C = 2 \quad \therefore\ C = 1$

$$\therefore\ \bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-10}{s^2-2s+10}\right\}$$

▶

Now we have to find the inverse transforms to obtain x. The first term $\dfrac{1}{s-2}$ is easy enough, but what of $\dfrac{s-10}{s^2-2s+10}$? The denominator will not factorise into simple linear factors; therefore we complete the square in the denominator and write it as

$$\frac{s-10}{s^2-2s+10}=\frac{s-10}{(s-1)^2+9}$$

and then we improve this still further and write it in the form $\dfrac{(s-1)-9}{(s-1)^2+9}$. We are quite happy with this, for $\dfrac{s-1}{(s-1)^2+9}$ is merely $\dfrac{s}{s^2+9}$ with s replaced by $(s-1)$, which indicates an extra factor e^t in the final function of t (Theorem 1).

So $\quad \bar{x}=\dfrac{1}{10}\left\{\dfrac{1}{s-2}-\dfrac{s-1}{(s-1)^2+9}+\dfrac{9}{(s-1)^2+9}\right\}$

$\therefore\ x=$ ....

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$$\boxed{x=\frac{1}{10}\{e^{2t}-e^t\cos 3t+3e^t\sin 3t\}}$$

Just try one more like this one

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Example 5

Solve $\ddot{x}+\dot{x}+x=e^{-t}$ given that at $t=0$, $x=0$ and $\dot{x}=1$. We find the expression for $\bar{x}$ as before.

$\bar{x}=$ ....

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$$\boxed{\bar{x}=\frac{s+2}{(s+1)(s^2+s+1)}}$$

Because $\left(s^2\bar{x}-sx_0-x_1\right)+(s\bar{x}-x_0)+\bar{x}=\dfrac{1}{s+1}$ where $x_0=0$ and $x_1=1$ so that

$$s^2\bar{x}-1+s\bar{x}+\bar{x}=\frac{1}{s+1}$$

therefore

$$\bar{x}\left(s^2+s+1\right)=1+\frac{1}{s+1}=\frac{s+2}{s+1}$$

giving

$$\bar{x}=\frac{s+2}{(s+1)(s^2+s+1)}$$

Expressing this in partial fractions

$\bar{x}=$ ....

Evaluate the coefficients

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$$\boxed{\bar{x} = \frac{1}{s+1} + \frac{s-1}{s^2+s+1}}$$

Because

$$\bar{x} = \frac{s+2}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$$

so that

$$s+2 = A\left(s^2+s+1\right) + (Bs+C)(s+1)$$

Put $s+1=0$, that is $s=-1$ then

$$\begin{array}{lll} & 1 = A(1-1+1) & \text{so that } A = 1 \\ [s^2] & 0 = A + B & \text{so that } B = -1 \\ [\text{CT}] & 2 = A + C & \text{so that } C = 1 \end{array}$$

Therefore

$$\bar{x} = \frac{1}{s+1} - \frac{s-1}{s^2+s+1}$$

Completing the squares in the second term gives

$$\frac{s-1}{s^2+s+1} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{s-1}{s^2+s+1} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

Because

$$\begin{aligned} \frac{s-1}{s^2+s+1} &= \frac{s-1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} \\ &= \frac{s+\frac{1}{2}-\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \end{aligned}$$

so that

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\bar{x} = \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

and so

$$x = \ldots\ldots\ldots\ldots$$

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$$x = e^{-t} - e^{-t/2}\cos\frac{\sqrt{3}t}{2} + \sqrt{3}e^{-t/2}\sin\frac{\sqrt{3}t}{2}$$

Before we leave this topic, the same general approach can be employed for solving simultaneous differential equations.

Let us see an example in the next frame

Simultaneous differential equations

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Example 1

Solve the pair of simultaneous equations

$$\dot{y} - x = e^{t}$$
$$\dot{x} + y = e^{-t}$$

given that at $t = 0$, $x = 0$ and $y = 0$.

(a) We first express both equations in Laplace transforms.

$$(s\bar{y} - y_0) - \bar{x} = \frac{1}{s-1}$$
$$(s\bar{x} - x_0) + \bar{y} = \frac{1}{s+1}$$

(b) Then we insert the initial conditions, $x_0 = 0$ and $y_0 = 0$.

$$\therefore \left.\begin{aligned} s\bar{y} - \bar{x} &= \frac{1}{s-1} \\ s\bar{x} + \bar{y} &= \frac{1}{s+1} \end{aligned}\right\} \qquad (1)$$

(c) We now solve these for $\bar{x}$ and $\bar{y}$ by the normal algebraic method. Eliminating $\bar{y}$ we have

$$s\bar{y} - \bar{x} = \frac{1}{s-1}$$
$$s\bar{y} + s^2\bar{x} = \frac{s}{s+1}$$
$$\therefore \quad (s^2+1)\bar{x} = \frac{2}{s+1} - \frac{1}{s-1} = \frac{s^2-2s-1}{(s+1)(s-1)}$$
$$\therefore \quad \bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)}$$

Representing this in partial fractions gives ....

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$$\boxed{\bar{x} = -\frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}}$$

Because

$$\bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore\ s^2-2s-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (s-1)(s+1)(Cs+D)$$

Putting $s = 1$ and $s = -1$ gives $A = -\frac{1}{2}$ and $B = -\frac{1}{2}$.
Comparing coefficients of s^3 and the constant terms gives $C = 1$ and $D = 1$.

$$\therefore\ \bar{x} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{s+1}{s^2+1}$$

$\therefore\ x = \ldots\ldots\ldots\ldots$

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$$\boxed{x = -\tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-t} + \cos t + \sin t}$$

We now revert to equations (1) and eliminate $\bar{x}$ to obtain $\bar{y}$ and hence y, in the same way. Do this on your own.

$y = \ldots\ldots\ldots\ldots$

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$$\boxed{y = \tfrac{1}{2}e^{t} + \tfrac{1}{2}e^{-t} - \cos t + \sin t}$$

Here is the working.

$$\left.\begin{aligned} s^2\bar{y} - s\bar{x} &= \frac{s}{s-1} \\ \bar{y} + s\bar{x} &= \frac{1}{s+1} \end{aligned}\right\}$$

$$\therefore\ (s^2+1)\bar{y} = \frac{s}{s-1} + \frac{1}{s+1} = \frac{s^2+2s-1}{(s-1)(s+1)}$$

$$\therefore\ \bar{y} = \frac{s^2+2s-1}{(s-1)(s+1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore\ s^2+2s-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (s-1)(s+1)(Cs+D)$$

Putting $s = 1$ and $s = -1$ gives $A = \frac{1}{2}$ and $B = \frac{1}{2}$.
Equating coefficients of s^3 and the constant terms gives $C = -1$ and $D = 1$.

$$\therefore\ \bar{y} = \frac{1}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore\ y = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - \cos t + \sin t$$

So the results are

$$x = -\frac{1}{2}(e^t + e^{-t}) + \sin t + \cos t = \sin t + \cos t - \cosh t$$

$$y = \frac{1}{2}(e^t + e^{-t}) + \sin t - \cos t = \sin t - \cos t + \cosh t$$

$$\therefore\ x = \sin t + \cos t - \cosh t; \quad y = \sin t - \cos t + \cosh t$$

Simultaneous equations are all solved in much the same way. Here is another.

Example 2

Solve the equations

$$2\dot{y} - 6y + 3x = 0$$

$$3\dot{x} - 3x - 2y = 0$$

given that at $t = 0$, $x = 1$ and $y = 3$.

Expressing these in Laplace transforms, we have

. . . .

. . . .

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$$\boxed{\begin{array}{l} 2(s\bar{y} - y_0) - 6\bar{y} + 3\bar{x} = 0 \\ 3(s\bar{x} - x_0) - 3\bar{x} - 2\bar{y} = 0 \end{array}}$$

Then we insert the initial conditions and simplify, obtaining

. . . .

. . . .

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$$\boxed{\begin{array}{ll} 3\bar{x} + (2s - 6)\bar{y} = 6 & (1) \\ (3s - 3)\bar{x} - 2\bar{y} = 3 & (2) \end{array}}$$

(a) *To find* $\bar{x}$

(1)	$3\bar{x} + (2s - 6)\bar{y} = 6$
(2) × $(s - 3)$	$(s - 3)(3s - 3)\bar{x} - (2s - 6)\bar{y} = 3(s - 3)$
Adding,	$[(s - 3)(3s - 3) + 3]\bar{x} = 3s - 9 + 6$
	$\therefore\ (3s^2 - 12s + 12)\bar{x} = 3s - 3$
	$(s^2 - 4s + 4)\bar{x} = s - 1$

$$\therefore\ \bar{x} = \frac{s - 1}{(s - 2)^2} \equiv \frac{A}{s - 2} + \frac{B}{(s - 2)^2} = \frac{A(s - 2) + B}{(s - 2)^2}$$

$$\therefore\ s - 1 = A(s - 2) + B \quad \text{giving} \quad A = 1 \quad \text{and} \quad B = 1$$

$$\therefore\ \bar{x} = \frac{1}{s - 2} + \frac{1}{(s - 2)^2} \qquad \therefore\ x = e^{2t} + te^{2t}$$

(b) Going back to equations (1) and (2), we can find y.

$y =$ . . . .

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$$\boxed{y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}}$$

Because, eliminating $\bar{x}$ we get

$$\bar{y} = \frac{6s-9}{2(s-2)^2} \equiv \frac{1}{2}\left\{\frac{A}{s-2} + \frac{B}{(s-2)^2}\right\} = \frac{1}{2}\left\{\frac{A(s-2)+B}{(s-2)^2}\right\}$$

$$\therefore\; 6s - 9 = A(s-2) + B \qquad \therefore\; A = 6;\; B = 3$$

$$\therefore\; \bar{y} = \frac{1}{2}\left\{\frac{6}{s-2} + \frac{3}{(s-2)^2}\right\} \qquad \therefore\; y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}$$

Simultaneous second-order equations are solved in like manner. Again, with all these solutions it is a worthwhile exercise to substitute the solution back into the differential equation to verify that the solution is correct.

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Example 3

If x and y are functions of t, solve the equations

$$\ddot{x} + 2x - y = 0$$
$$\ddot{y} + 2y - x = 0$$

given that at $t = 0$, $x_0 = 4$; $y_0 = 2$; $x_1 = 0$; $y_1 = 0$.

We start off as usual with $(s^2\bar{x} - sx_0 - x_1) + 2\bar{x} - \bar{y} = 0$

and $(s^2\bar{y} - sy_0 - y_1) + 2\bar{y} - \bar{x} = 0$

Inserting the initial conditions, we have

$$s^2\bar{x} - 4s + 2\bar{x} - \bar{y} = 0$$
$$s^2\bar{y} - 2s + 2\bar{y} - \bar{x} = 0$$

Simplifying these we can eliminate $\bar{y}$ to obtain $\bar{x}$ and hence x.

$x = \dots\dots\dots\dots$

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$$\boxed{x = 3\cos t + \cos\left(\sqrt{3}t\right)}$$

Because

$$(s^2+2)\bar{x} - \bar{y} = 4s \qquad (1)$$

$$-\bar{x} + (s^2+2)\bar{y} = 2s \qquad (2)$$

Eliminating $\bar{y}$ and simplifying gives

$$\bar{x} = \frac{4s^3+10s}{(s^2+1)(s^2+3)}$$

$$\therefore\ \bar{x} = \frac{4s^3+10s}{(s^2+1)(s^2+3)} \equiv \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+3}$$

$$\therefore\ 4s^3+10s = \left(s^2+3\right)(As+B) + \left(s^2+1\right)(Cs+D)$$

Equating coefficients of like powers of s

$[s^3]$ $\quad 4 = A + C \quad \therefore\ A + C = 4$

[CT] $\quad 0 = 3B + D \quad \therefore\ 3B + D = 0$

Putting $s = 1$, $\quad 14 = 4A + 4B + 2C = 2D \quad \therefore\ 2A + 2B + C + D = 7$

Putting $s = -1$ $\quad -14 = -4A + 4B - 2C + 2D \quad \therefore\ 2A - 2B + C - D = 7$

Putting $C = 4 - A$ and $D = -3B$ in the last two leads to

$A = \ldots\ldots\ldots\ldots$; $\quad B = \ldots\ldots\ldots\ldots$;

$C = \ldots\ldots\ldots\ldots$; $\quad D = \ldots\ldots\ldots\ldots$

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$$\boxed{A = 3;\quad B = 0;\quad C = 1;\quad D = 0}$$

$$\therefore\ \bar{x} = \frac{3s}{s^2+1} + \frac{s}{s^2+3}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$\boxed{x = 3\cos t + \cos\left(\sqrt{3}t\right)}$$

To find y we could return to equations (1) and (2) and repeat the process, eliminating $\bar{x}$ so as to obtain $\bar{y}$ and hence y.

But always keep an eye on the original equations, the first of which is

$$\ddot{x} + 2x - y = 0$$

Therefore, in this particular case, $y = \ddot{x} + 2x$.

So all we have to do is to differentiate x twice and substitute

$$x = 3\cos t + \cos\left(\sqrt{3}t\right)$$

$$\dot{x} = -3\sin t - \sqrt{3}\sin\left(\sqrt{3}t\right)$$

$$\ddot{x} = -3\cos t - 3\cos\left(\sqrt{3}t\right)$$

$$\therefore\ y = -3\cos t - 3\cos\left(\sqrt{3}t\right) + 6\cos t + 2\cos\left(\sqrt{3}t\right)$$

$$\therefore\ y = 3\cos t - \cos\left(\sqrt{3}t\right)$$

which is a good deal quicker.

So, as we have seen, the method of solving differential equations by Laplace transforms follows a general routine.

(a) Express the equation in Laplace transforms
(b) Insert the initial conditions
(c) Simplify to obtain the transform of the solution
(d) Rewrite the final transform in partial fractions
(e) Determine the inverse transforms

and, by now, you are fully aware of the importance of *partial fractions*!

That brings us to the end of this particular Programme. We shall continue our study of Laplace transforms in the next Programme. Meanwhile, be sure you are familiar with the items listed in the **Revision summary** that follows, and respond to the questions in the **Can You?** checklist. You will then have no difficulty with the **Test Exercise** and the **Further problems** provide additional practice.

Revision summary 2

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1 *Laplace transform* $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$.

2 *Table of transforms*

$f(t)$	$L\{f(t)\} = F(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$ (n a positive integer)

3 *Linearity of the Laplace transform*

(a) The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(b) The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is

$$L\{kf(t)\} = kL\{f(t)\}$$

4 *Theorem 1* First shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$.

5 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$.

6 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

provided that $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.

7 *Inverse transform*

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\} = f(t)$.

8 *Rules of partial fractions*

(a) The numerator must be of lower degree than the denominator. If not, divide out.

(b) Factorise the denominator into its prime factors.

(c) A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.

(d) A repeated factor $(s+a)^2$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}$.

(e) Similarly $(s+a)^3$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$.

(f) A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$.

(g) A repeated quadratic factor $(s^2+ps+q)^2$ gives

$$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

9 *The 'cover up' rule*

The 'cover up' rule often enables the values of the constant coefficients to be written down almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors.

10 *Table of inverse transforms*

$F(s)$	$f(t)$	
$\dfrac{a}{s}$	a	
$\dfrac{1}{s+a}$	e^{-at}	
$\dfrac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$	
$\dfrac{a}{s^2+a^2}$	$\sin at$	
$\dfrac{s}{s^2+a^2}$	$\cos at$	
$\dfrac{a}{s^2-a^2}$	$\sinh at$	
$\dfrac{s}{s^2-a^2}$	$\cosh at$	

By the first shift theorem

If $F(s)$ is the Laplace transform of $f(t)$
then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$. ▶

11 *Laplace transforms of derivatives*

$$L\{x\} = \bar{x}$$

$$L\left\{\frac{dx}{dt}\right\} = L\{\dot{x}\} = s\bar{x} - x_0$$

$$L\left\{\frac{d^2x}{dt^2}\right\} = L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1 \text{ etc.}$$

where x_0 = value of x at $t = 0$

x_1 = value of $\dfrac{dx}{dt}$ at $t = 0$, etc.

12 *Solution of differential equations*

(a) Rewrite the equation in terms of Laplace transforms.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.
(d) Express the transform in standard forms by partial fractions.
(e) Determine the inverse transforms to obtain the particular solution.

13 *Simultaneous differential equations*

Convert the simultaneous differential equations into simultaneous algebraic equations by taking the Laplace transform of each equation in turn. Insert the initial values. Solve the simultaneous algebraic equations in the usual manner and take the inverse Laplace transform of the algebraic solutions to find the solutions to the simultaneous differential equations.

Can You?

Checklist 2

91

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:

Frames

- Obtain the Laplace transforms of simple standard expressions? — Frames 1 to 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the first shift theorem to find the Laplace transform of a simple expression multiplied by an exponential? — Frames 15 to 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the Laplace transform of a simple expression multiplied or divided by a variable? — Frames 18 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use partial fractions to find the inverse Laplace transform? 27 to 41

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the 'cover up' rule? 42 to 45

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transforms of derivatives to solve differential equations? 46 to 78

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transform to solve simultaneous differential equations? 79 to 89

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 2

92

1 Determine the Laplace transforms of the following functions.

(a) $3e^{-4t} - 5e^{4t}$ (b) $\sin 4t + \cos 4t$ (c) $t^3 + 2t^2 - t + 4$

(d) $e^{-2t}\cos 5t$ (e) $t\sin 3t$ (f) $\dfrac{e^{-t} - e^{-2t}}{t}$.

2 Determine the inverse transforms of the following.

(a) $\dfrac{s-5}{(s-3)(s-4)}$ (b) $\dfrac{s^2+3s-7}{(s-1)(s^2+2)}$

(c) $\dfrac{s^2-3s-4}{(s-3)(s-1)^2}$ (d) $\dfrac{2s^2-6s-1}{(s-3)(s^2-2s+5)}$.

3 Solve the following equations by Laplace transforms.

(a) $\dfrac{dx}{dt} + 3x = e^{-2t}$ given that $x = 2$ when $t = 0$

(b) $3\dot{x} - 6x = \sin 2t$ given that $x = 1$ when $t = 0$

(c) $\ddot{x} - 7\dot{x} + 12x = 2$ given that at $t = 0$, $x = 1$ and $\dot{x} = 5$

(d) $\ddot{x} - 2\dot{x} + x = te^t$ given that at $t = 0$, $x = 1$ and $\dot{x} = 0$.

4 Solve the following pair of simultaneous equations where x and y are functions of t and given that at $t = 0$, $x = 4$ and $y = -1$.

$$\dot{x} + \dot{y} + x + 2y = e^{-3t}$$
$$\dot{x} + 3x + 5y = 5e^{-2t}$$

Further problems 2

93

1 Determine the Laplace transforms of the following functions.

(a) $e^{4t}\cos 2t$ (b) $t\sin 2t$ (c) $t^3 + 4t^2 + 5$

(d) $e^{3t}(t^2 + 4)$ (e) $t^2\cos t$ (f) $\dfrac{\sinh 2t}{t}$.

2 Determine the inverse transforms of the following.

(a) $\dfrac{2s-6}{(s-2)(s-4)}$ (b) $\dfrac{5s-8}{s(s-4)}$ (c) $\dfrac{s^2-2s+3}{(s-2)^3}$

(d) $\dfrac{2-11s}{(s-2)(s^2+2s+2)}$ (e) $\dfrac{s}{(s^2+1)(s^2+4)}$ (f) $\dfrac{s-5}{s^2+4s+20}$.

In Questions 3 to 11, solve the equations by Laplace transforms.

3 $\dot{x}-4x=8$ at $t=0,\ x=2$.

4 $3\dot{x}-4x=\sin 2t$ at $t=0,\ x=\frac{1}{3}$.

5 $\ddot{x}-2\dot{x}+x=2(t+\sin t)$ at $t=0,\ x=6,\ \dot{x}=5$.

6 $\ddot{x}-6\dot{x}+8x=e^{3t}$ at $t=0,\ x=0,\ \dot{x}=2$.

7 $\ddot{x}+9x=\cos 2t$ at $t=0,\ x=1,\ \dot{x}=3$.

8 $\ddot{x}-2\dot{x}+5x=e^{2t}$ at $t=0,\ x=0,\ \dot{x}=1$.

9 $\ddot{x}+4\dot{x}+4x=t^2+e^{-2t}$ at $t=0,\ x=\frac{1}{2},\ \dot{x}=0$.

10 $\ddot{x}+8\dot{x}+32x=32\sin 4t$ at $t=0,\ x=\dot{x}=0$.

11 $\ddot{x}+25x=10(\cos 5t-2\sin 5t)$ at $t=0,\ x=1,\ \dot{x}=2$.

In Questions 12 to 17, solve the pairs of simultaneous equations by Laplace transforms.

12 $\left.\begin{aligned}\dot{y}+3x&=e^{-2t}\\ \dot{x}-3y&=e^{2t}\end{aligned}\right\}$ at $t=0,\ x=y=0$.

13 $\left.\begin{aligned}4\dot{x}-2\dot{y}+10x-5y&=0\\ \dot{y}-18x+15y&=10\end{aligned}\right\}$ at $t=0,\ y=4,\ x=2$.

14 $\left.\begin{aligned}\dot{x}-2\dot{y}-3x+6y&=12\\ 3\dot{y}+5x+2y&=16\end{aligned}\right\}$ at $t=0,\ x=12,\ y=8$.

15 $\left.\begin{aligned}2\dot{x}+3\dot{y}+7x&=14t+7\\ 5\dot{x}-3\dot{y}+4x+6y&=14t-14\end{aligned}\right\}$ at $t=0,\ x=y=0$.

16 $\left.\begin{aligned}2\dot{x}+2x+3\dot{y}+6y&=56e^{t}-3e^{-t}\\ \dot{x}-2x-\dot{y}-3y&=-21e^{t}-7e^{-t}\end{aligned}\right\}$ at $t=0,\ x=8,\ y=3$.

17 $\left.\begin{aligned}\ddot{x}-\ddot{y}+x-y&=5e^{2t}\\ 2\dot{x}-\dot{y}+y&=0\end{aligned}\right\}$ at $t=0,\ x=1,\ y=2,\ \dot{x}=0$.

18 Find an expression for x in terms of t, given that

$\ddot{y}-\dot{x}+2x=10\sin 2t$

$\dot{y}+2y+x=0$ and when $t=0,\ x=y=0$.

19 If $\ddot{x}+8x+2y=24\cos 4t$
and $\ddot{y}+2x+5y=0$
and at $t=0,\ x=y=0,\ \dot{x}=1,\ \dot{y}=2$, determine an expression for y in terms of t.

20 Solve completely, the pair of simultaneous equations

$5\ddot{x}+12\ddot{y}+6x=0$

$5\ddot{x}+16\ddot{y}+6y=0$

given that, at $t=0,\ x=\frac{7}{4},\ y=1,\ \dot{x}=0,\ \dot{y}=0$.